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Re-Entry Heat Conduction of a Finite Slab with a Nonconstant Thermal Conductivity

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The intense temperatures at the surface of a space vehicle may, in some instances, alter the thermal conductivity of the layers of the skin below the surface. This analysis presents, as a first approximation to the problem, a closed-form solution for the case of a thermal conductivity that varies linearly with distance below the surface. A convective heat input at the surface which is exponential in time is assumed. The results should apply for the initial phases of entry in which the entry velocity and angle are nearly constant. Author

Nomenclature

a = slope of thermal conductivity curve
B = constant that depends on entry velocity and angle
c = specific heat of material
C_1,C_2 = arbitrary constants
i = $-1^{1/2}$, imaginary unit
I_0,I_1 = modified Bessel functions of the first kind of order, zero and one, respectively
J_0,J_1 = Bessel functions of the first kind of order, zero and one, respectively
K = thermal conductivity
K_0,K_1 = modified Bessel functions of the second kind of order, zero and one, respectively
s = variable in the Laplace transform
t = time measured from entry
$T,\bar{T}$ = temperature and transformed temperature, respectively
V = entry velocity
x = distance normal to surface
Y_0,Y_1 = Bessel functions of the second kind of order, zero and one, respectively
γ = entry angle
θ = $\rho c/a^2$
ρ = density of material
τ = thickness of material
ω = constant that depends on entry velocity and angle

Subscripts

i = initial conditions
f = final conditions

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THE solution to the problem of one-dimensional heat conduction through a skin with constant thermal conductivity for the initial phases of a re-entry in which the velocity and entry angle are nearly constant and for which the convective heat rate is the dominant input can be found in Ref. 1. The present analysis intends to take the basic

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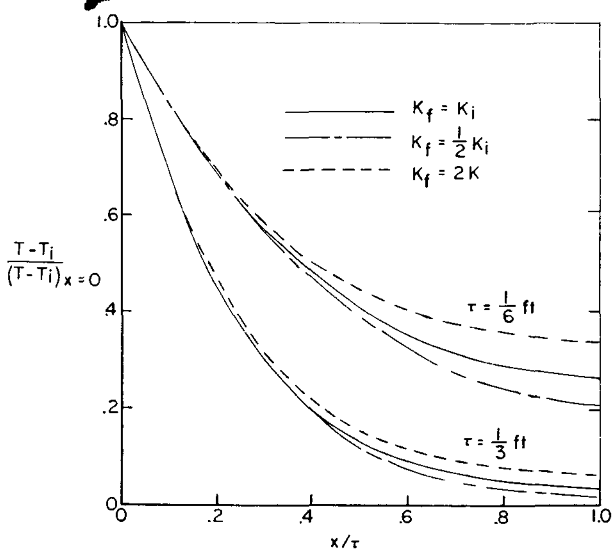

Fig. 1 Temperature drop across skins having constant and nonconstant thermal conductivities.

problem in Ref. 1 and extend it to the case of a vehicle having a skin whose thermal conductivity varies linearly with depth below the surface. Such a solution might serve as a first approximation to a situation in which the thermal conductivity of the layers of the skin below the surface are altered by the intense temperature at the surface.

The temperature history through the skin can be obtained from the one-dimensional heat-conduction equation which is

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial T}{\partial x}\right) \tag{1}$$

The boundary conditions will be taken to be the same as those given in Ref. 1, where it is assumed that, at the surface, the heat input is given by the convective heat rate that during the initial phase of entry can be represented by an exponential function in time. The back side of the skin is insulated, and initially the skin is at a constant temperature throughout. These conditions stated mathematically are

$$\left.\begin{aligned} \frac{\partial T}{\partial x}_{x=0} &= B(V,\gamma)\exp[\omega(V,\gamma)t] \\ \frac{\partial T}{\partial x}_{x=\tau} &= 0 \\ T(x,0) &= T_i \end{aligned}\right\} \tag{2}$$

$B(V,\gamma)$ and $\omega(V,\gamma)$ are constants that depend on the entry velocity and angle.

$$T(x,t) = T_i + \frac{BK_i}{\rho c \omega \tau} + \frac{B}{a}\left(\frac{K_i}{\omega\theta}\right)^{1/2} \times e^{\omega t} \left\{\frac{K_1[2(\theta\omega K_f)^{1/2}]I_0[2(\theta\omega K)^{1/2}] + I_1[2(\theta\omega K_f)^{1/2}]K_0[2(\theta\omega K)^{1/2}]}{I_1[2(\theta\omega K_i)^{1/2}]K_1[2(\theta\omega K_f)^{1/2}] - I_1[2(\theta\omega K_f)^{1/2}]K_1[2(\theta\omega K_i)^{1/2}]}\right\}$$
$$+ \sum_{m=1}^{\infty} \frac{Be^{-\beta_m^2 t}}{a\theta(\omega + \beta_m^2)\Omega_m}\{J_0[2\beta_m(0K)^{1/2}]Y_1[2\beta_m(\theta K_f)^{1/2}] - Y_0[2\beta_m(\theta K)^{1/2}]J_1[2\beta_m(\theta K_f)^{1/2}]\} \tag{12}$$

We will assume that the thermal conductivity varies with depth as

$$K(x) = K_i + ax \tag{3}$$

where

$$a = (K_f - K_i)/\tau \tag{4}$$

K_i, K_f, and τ are the initial and final values for the thermal conductivity and skin thickness, respectively.

Substitution of (3) into (1) gives

$$\rho c \frac{\partial T}{\partial t} = (K_i + ax)\frac{\partial^2 T}{\partial x^2} + a\frac{\partial T}{\partial x} \tag{5}$$

The solution of Eq. (5) will be obtained by the application of the Laplace transform. The transform of Eq. (5) gives

$$K\frac{d^2\bar{T}}{dK^2} + \frac{d\bar{T}}{dK} - \theta s \bar{T} = -\theta T_i \tag{6}$$

where $\bar{T}(x,s)$ is the transform of $T(x,t)$ and $\theta \equiv \rho c/a^2$.

The solution of (6) is

$$\bar{T}(x,s) = C_1 I_0[2(\theta s K)^{1/2}] + C_2 K_0[2(\theta s K)^{1/2}] + (T_i/s) \tag{7}$$

when I_0 and K_0 are the first and second kind of the modified Bessel functions of zero order.

The transformations of the boundary conditions (2) are

$$\left.\begin{aligned} \frac{d\bar{T}}{dx}_{x=0} &= \frac{B}{s-\omega} \\ \frac{d\bar{T}}{dx}_{x=\tau} &= 0 \\ \bar{T}(x,0) &= T_i/s \end{aligned}\right\} \tag{8}$$

Use of Eq. (8) in Eq. (7) gives, for the transformed temperature,

$$\bar{T}(x,s) = \frac{T_i}{s} + \frac{B}{a(s-\omega)}\left(\frac{K_i}{\theta s}\right)^{1/2} \times \left\{\frac{K_1[2(\theta s K_f)^{1/2}]I_0[2(\theta s K)^{1/2}] + I_1[2(\theta s K_f)^{1/2}]K_0[2(\theta s K)^{1/2}]}{I_1[2(\theta_i K_i)^{1/2}]K_1[2(\theta s K_f)^{1/2}] - I_1[2(\theta s K_f)^{1/2}]K_1[2(\theta s K_i)^{1/2}]}\right\} \tag{9}$$

The poles of $\bar{T}(x,s)$ are at $s = 0$, ω, and the roots of the denominator of the expression in braces in Eq. (9). These are all simple poles. The apparent branch point corresponding to the factor $(K_i/\theta s)^{1/2}$ actually provides an extra contribution to the residue of the pole at $s = 0$, because of the behavior of the expression in braces as $s \to 0$.

To find the poles associated with the expression in braces, we use the relations $I_0(x) = J_0(ix)$ and $K_0(x) = \frac{1}{2}\pi i[J_0(ix) + iY_0(ix)]$. These poles are then the roots of

$$J_1[2i(\theta s K_f)^{1/2}]Y_1[2i(\theta s K_i)^{1/2}] - J_1[2i(\theta s K_i)^{1/2}]Y_1[2i(\theta s K_f)^{1/2}] = 0 \tag{10}$$

J_1 and Y_1 are Bessel functions of the first and second kind of order one.

Let β_m be the roots of (10) such that $is^{1/2} = \beta_m$, or

$$s = -\beta_m^2 \qquad (m = 1, 2, \ldots) \tag{11}$$

The poles of $\bar{T}(x,s)$ are then at $s = 0$, ω, and $-\beta_m^2$. The values of β_m can be found in Ref. 2. The temperature is then given as

where

$$\Omega_m = J_1[2\beta_m(\theta K_f)^{1/2}]Y_0[2\beta_m(\theta K_i)^{1/2}] - J_0[2\beta_m(\theta K_i)^{1/2}]Y_1[2\beta_m(\theta K_f)^{1/2}] + \left(\frac{K_f}{K_i}\right)^{1/2}\{Y_1[2\beta_m(\theta K_i)^{1/2}]J_0[2\beta_m(\theta K_f)^{1/2}] - J_1[2\beta_m(\theta K_i)^{1/2}]Y_0[2\beta_m(\theta K_f)^{1/2}]\} \tag{13}$$

Some numerical computations based on Eq. (12) are shown in Figs. 1 and 2. In Fig. 1 a comparison is made of the stagnation-point temperature at any depth in the skin to that at the surface for the case of the thermal conductivity remaining constant, decreasing to one-half its initial value, and

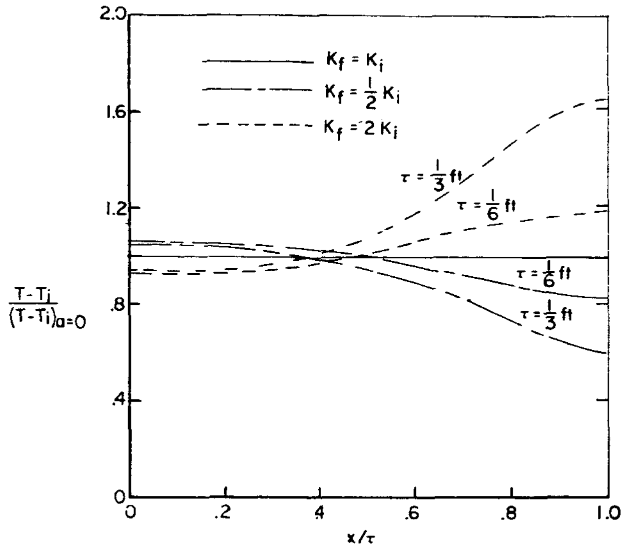

Fig. 2 Comparison of temperatures in skins having constant and nonconstant thermal conductivities.

increasing to twice its initial value. Figure 2 shows a comparison of the temperature in a skin having a nonconstant thermal conductivity to one for which the thermal conductivity is constant. The curves in Fig. 2 show a slight relief in temperature up to about the middle of the skin for the case of the thermal conductivity increasing with depth. However, for the back half of the skin, this variation in thermal conductivity shows a rapid increase in temperature. On the other hand, there is a slight temperature rise in the first half of the skin for the case of decreasing thermal conductivity with an appreciable decrease in temperature for the back half of the skin for this variation in thermal conductivity.

For these figures an entry velocity of 20,000 fps at 400,000 ft and an entry angle of $-20°$ was used. The initial value of the thermal conductivity used was 0.548 Btu/ft-sec-°F which corresponds to electrolytic copper at 1000°F. The results shown are time independent after about 10 sec.

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